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The unexpected flight of the electron in a classical hydrogen-like atom

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Abstract
We study the behaviour of a particle in an attractive Coulomb potential according to the nonrelativistic approximation of the Lorentz–Dirac equation, that is the equation describing the motion of a charged point particle in classical electrodynamics when radiation reaction is taken into account. We prove that for any choice of the initial data the particle always eventually escapes to infinity. Its acceleration either vanishes asymptotically, corresponding to a scattering process, or increases exponentially fast with time, with a so-called 'runaway' behaviour. No bound states of any type are possible.

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1. Introduction

Let us consider a point particle of mass m and electric charge $-e$, moving in the attractive Coulomb potential generated by a fixed point charge Ze. In classical electrodynamics, when the effect of radiation reaction is taken into account, the motion of the particle obeys the Lorentz–Dirac equation [1], which in the situation considered and in three-dimensional notation can be written as

$$\frac{6\pi m}{e^2}\ddot{\boldsymbol{x}} - \dddot{\boldsymbol{x}} - \dot{\boldsymbol{x}}(\ddot{t}^2 - \ddot{\boldsymbol{x}}^2) = -\frac{3Z}{2}\frac{\boldsymbol{x}}{r^3}\dot{t} \qquad \dot{t}^2 - \dot{\boldsymbol{x}}^2 = 1 \tag{1}$$

where the vector $\boldsymbol{x}$ represents the coordinates of the particle with respect to the fixed centre of force, $r = |\boldsymbol{x}|$ and dots denote differentiation with respect to proper time s (units are such that the speed of light is equal to one).

In 1943 Eliezer [2] proved a quite surprising result about the particular solutions of equation (1) for which the motion takes place on a straight line. He found that, despite the attractive character of the Coulomb potential, it is impossible for a particle to fall into the centre of force. An incoming particle always inverts its direction of motion at a finite distance from the singularity, and then starts moving away in a self-accelerating way, in such a way that its velocity asymptotically approaches the speed of light [3]. Such unphysical 'runaway' behaviour was already known to occur for solutions of the Lorentz–Dirac equation even in the absence of an external potential, e.g., for the motion of a free particle or of a particle disturbed by an electromagnetic pulse [1]. In those cases, for assigned initial values of position and velocity, physically acceptable solutions with constant final momentum can only be obtained by a suitable choice of the initial acceleration. However, the fact that runaway flight, instead of collision, is the only possible outcome of the one-dimensional Coulomb problem whenever the initial velocity is directed towards the centre of force, appears as a remarkable contradiction to elementary physical intuition.

More recently, Eliezer's results have been partially extended to the full three-dimensional case by Carati [4], who has considered the mathematically simpler equation [5] to which (1) reduces in the nonrelativistic limit. This equation is

$$\frac{6\pi m}{e^2}\ddot{\boldsymbol{x}} - \dddot{\boldsymbol{x}} = -\frac{3Z}{2}\frac{\boldsymbol{x}}{r^3} \tag{2}$$

where dots now denote differentiation with respect to the laboratory time t. Carati has proved that there do not exist solutions of (2) for which the particle falls into the centre of force (i.e. $r \to 0$) either at a finite or an infinite time. This result, which was already conjectured by Eliezer for the relativistic case [2], contradicts the well-known heuristic argument [6] according to which the electron of an hydrogen-like atom should fall into the nucleus in a short time by spiralling inwards as a result of energy loss by radiation.

However, apart from this negative result, no other information was available up to now about the possible behaviour of the particle for $t \to +\infty$ in the three-dimensional case. In particular, no definite answer still existed about the possible existence of bounded solutions [4]. In this paper, dealing again with the nonrelativistic case, we provide an answer to this question by proving that actually the particle, for any choice of the initial data, eventually escapes to infinity (i.e. $\lim_{t\to+\infty} r = \infty$), either with runaway behaviour or with vanishing asymptotic acceleration.

2. Mathematical results

After the rescaling $t \to \xi t$, with $\xi = e^2/6\pi m$ and $\boldsymbol{x} \to \chi\boldsymbol{x}$, with $\chi = (3Z/2)^{1/3}\xi$, equation (2) becomes

$$\dddot{\boldsymbol{x}} = \ddot{\boldsymbol{x}} + \frac{\boldsymbol{x}}{r^3}. \tag{3}$$

The present section is devoted to a mathematical analysis of the properties of the solutions of the above equation. We recall that equation (3) shares with (1) the feature of admitting self-accelerating solutions. To see this, just note that when the Coulomb term vanishes, that is for $r \to \infty$, equation (3) obviously admits solutions of the form $\ddot{\boldsymbol{x}}(t) = \ddot{\boldsymbol{x}}_0\,\mathrm{e}^t$. The main difference, with respect to the relativistic case, is that now the limit of the velocity is infinite instead of one. We shall be using the notation $\boldsymbol{v} = \dot{\boldsymbol{x}}, \boldsymbol{a} = \ddot{\boldsymbol{x}}, v = |\boldsymbol{v}|, a = |\boldsymbol{a}|$. Since equation (3) is of third order with respect to time, initial data include the values of $\boldsymbol{x}, \boldsymbol{v}$ and $\boldsymbol{a}$ at a given time t_0 . The fact that the particle cannot fall into the singularity in a finite time implies that for any initial data the solution is univocally defined and well behaved on the entire time interval $[t_0, +\infty)$.

We begin by giving some useful lemmas.

Lemma 1. *If $y \in \mathbb{R}$ and $f : [\bar{t}, +\infty) \to \mathbb{R}$ is a differentiable function such that $f(\bar{t}) < y < \limsup_{s\to+\infty} f(s)$, then there exists $t > \bar{t}$ such that $f(t) = y$, $\dot{f}(t) \geqslant 0$.*

Proof. Since f is continuous and $\limsup_{s\to+\infty} f(s) > y$, $f^{-1}([y, +\infty))$ is a closed nonempty subset of $[\bar{t}, +\infty)$, and therefore it has a minimum. Then the thesis is easily obtained with $t = \min f^{-1}([y, +\infty))$. □

Lemma 2. *If $\boldsymbol{x} : [\bar{t}, +\infty) \to \mathbb{R}^N$ is a twice continuously differentiable function such that $\int_{\bar{t}}^{+\infty} a^2\,\mathrm{d}t < +\infty$ and $r < M$ for every $t \geqslant \bar{t}$, where M is a positive constant, then* $\lim_{t\to+\infty} \boldsymbol{v} = 0$.

Proof. Let us suppose that $\limsup_{t\to+\infty} v > 0$ and take $\bar{v}$ such that $0 < \bar{v} < \limsup_{t\to+\infty} v$. For any arbitrarily large T, owing to the fact that $\int_{\bar{t}}^{+\infty} a^2\,\mathrm{d}t < +\infty$, one can take t_0 such that $\int_{t_0}^{+\infty} a^2\,\mathrm{d}t < \bar{v}^2/4T$. Then, from Schwarz's inequality it follows that for $t > t_0$

$$\int_t^{t+T} a\,\mathrm{d}t' \leqslant \left(\int_t^{t+T} a^2\,\mathrm{d}t'\right)^{1/2} \left(\int_t^{t+T} \mathrm{d}t'\right)^{1/2} < \sqrt{\frac{\bar{v}^2}{4T}}\sqrt{T} = \frac{\bar{v}}{2}.$$

If we now take $t_1 > t_0$ such that $v(t_1) > \bar{v}$, then for all $t \in [t_1, t_1 + T]$ we have $|\boldsymbol{v}(t) - \boldsymbol{v}(t_1)| \leqslant \int_{t_1}^{t} a\,\mathrm{d}t' < \bar{v}/2$. Introducing the unit vector $\boldsymbol{n} = \boldsymbol{v}(t_1)/v(t_1)$, we have then $v(t_1) - \boldsymbol{n}\cdot\boldsymbol{v}(t) = \boldsymbol{n}\cdot[\boldsymbol{v}(t_1) - \boldsymbol{v}(t)] < \bar{v}/2$, whence $\boldsymbol{n}\cdot\boldsymbol{v}(t) > v(t_1) - \bar{v}/2 > \bar{v}/2$. It follows that

$$|\boldsymbol{x}(t_1 + T) - \boldsymbol{x}(t_1)| \geqslant \boldsymbol{n}\cdot[\boldsymbol{x}(t_1 + T) - \boldsymbol{x}(t_1)] = \int_{t_1}^{t_1+T} \boldsymbol{n}\cdot\boldsymbol{v}(t)\,\mathrm{d}t > \frac{\bar{v}T}{2}$$

which implies that the trajectory is unbounded, owing to the arbitrariness of T. One can conclude that if the trajectory is bounded then $\limsup_{t\to+\infty} v = 0$, or equivalently $\lim_{t\to+\infty} \boldsymbol{v} = 0$. □

Lemma 3. *If $f : [t_1, t_2] \to \mathbb{R}$ is a continuously differentiable function such that $\dot{f} - f \geqslant A$ for every $t \in [t_1, t_2]$, where A is a positive constant, then $\int_{t_1}^{t_2} f^2\,\mathrm{d}t \geqslant A^2[T - 2\tanh(T/2)]$, with $T = t_2 - t_1$.*

Proof. Introducing the function $h = \dot{f} - f = \mathrm{e}^t\,\mathrm{d}/\mathrm{d}t\,(\mathrm{e}^{-t} f) \geqslant A$, we have for $t_1 \leqslant t \leqslant s \leqslant t_2$

$$\mathrm{e}^{-s} f(s) - \mathrm{e}^{-t} f(t) = \int_t^s \mathrm{e}^{-u} h(u)\,\mathrm{d}u \geqslant A(\mathrm{e}^{-t} - \mathrm{e}^{-s})$$

whence it follows that

$$f(s) \geqslant A(\mathrm{e}^{s-t} - 1) \geqslant 0 \qquad \text{for } s \geqslant t,\ f(t) \geqslant 0 \tag{4a}$$

$$f(s) \leqslant A(\mathrm{e}^{s-t} - 1) \leqslant 0 \qquad \text{for } s \leqslant t,\ f(t) \leqslant 0. \tag{4b}$$

Let us now take t in the following way:

(i) $t = t_1$ if $f(t_1) \geqslant 0$;
(ii) $t = t_2$ if $f(t_1) < 0$, $f(t_2) \leqslant 0$;
(iii) $t \in (t_1, t_2)$ such that $f(t) = 0$ if $f(t_1) < 0$, $f(t_2) > 0$.

By applying (4*a*) and (4*b*) it is then easy to see that

$$[f(s)]^2 \geqslant A^2(\mathrm{e}^{s-t} - 1)^2 \qquad \text{for all } s \in [t_1, t_2].$$

It follows that $\int_{t_1}^{t_2} f^2\,\mathrm{d}s \geqslant A^2 g(t)$, where

$$g(t) = \int_{t_1}^{t_2} (\mathrm{e}^{s-t} - 1)^2\,\mathrm{d}s = t_2 - t_1 - 2\,\mathrm{e}^{-t}(\mathrm{e}^{t_2} - \mathrm{e}^{t_1}) + \mathrm{e}^{-2t}\,\frac{\mathrm{e}^{2t_2} - \mathrm{e}^{2t_1}}{2}.$$

We can then write in all cases

$$\int_{t_1}^{t_2} f^2\,\mathrm{d}s \geqslant A^2 \min_{t\in[t_1,t_2]} g(t).$$

By equating to 0 the derivative of g one finds that the minimum is reached at $\tilde{t} = \ln[(\mathrm{e}^{t_1}+\mathrm{e}^{t_2})/2]$, so that

$$A^{-2}\int_{t_1}^{t_2} f^2\,\mathrm{d}s \geqslant g(\tilde{t}) = t_2 - t_1 - 2\tanh\frac{t_2-t_1}{2} \geqslant t_2 - t_1 - 2.$$

□

Following [4], we derive now from equation (3) a set of useful relations. We first recall that (3) can be rewritten in the form

$$\dot{\boldsymbol{a}} - \boldsymbol{a} = \mathrm{e}^t\frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{e}^{-t}\boldsymbol{a}(t)] = \frac{\boldsymbol{x}}{r^3}$$

which can be integrated to give

$$\boldsymbol{a}(t) = \mathrm{e}^{t-t_0}\left(\boldsymbol{a}(t_0) + \int_{t_0}^{t} \mathrm{e}^{t_0-s}\frac{\boldsymbol{x}(s)}{r^3(s)}\,\mathrm{d}s\right). \tag{5}$$

Introducing the function

$$E = \frac{v^2}{2} - \frac{1}{r} - \boldsymbol{a}\cdot\boldsymbol{v} \tag{6}$$

from (3) one has $\mathrm{d}E/\mathrm{d}t = -a^2$, or equivalently

$$E(t) = E(\bar{t}) - \int_{\bar{t}}^{t} a^2\,\mathrm{d}s. \tag{7}$$

The function E can be considered as a generalized particle energy which decreases with time as a result of radiation. Putting $F = \boldsymbol{a}\cdot\boldsymbol{x} - E$, from (3) it also follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{a}\cdot\boldsymbol{x}) = \dot{\boldsymbol{a}}\cdot\boldsymbol{x} + \boldsymbol{a}\cdot\boldsymbol{v} = F + \frac{v^2}{2} \tag{8}$$

$$\frac{\mathrm{d}F}{\mathrm{d}t} = F + \frac{v^2}{2} + a^2. \tag{9}$$

From (9) we obtain for $t > t_0$

$$F(t) = \mathrm{e}^{t-t_0}F(t_0) + \int_{t_0}^{t}\mathrm{d}s\,\mathrm{e}^{t-s}\left(\frac{v^2}{2} + a^2\right) \geqslant \mathrm{e}^{t-t_0}F(t_0)$$

so that using (8)

$$\begin{aligned}\boldsymbol{a}(t)\cdot\boldsymbol{x}(t) &\geqslant \boldsymbol{a}(t_0)\cdot\boldsymbol{x}(t_0) + \int_{t_0}^{t} F(s)\,\mathrm{d}s\\ &\geqslant \boldsymbol{a}(t_0)\cdot\boldsymbol{x}(t_0) + F(t_0)(\mathrm{e}^{t-t_0}-1) = E(t_0) + \mathrm{e}^{t-t_0}F(t_0).\end{aligned} \tag{10}$$

We also have

$$\frac{\mathrm{d}a^2}{\mathrm{d}t} = 2\boldsymbol{a}\cdot\dot{\boldsymbol{a}} = 2a^2 + 2\frac{\boldsymbol{a}\cdot\boldsymbol{x}}{r^3}$$

whence

$$a^2(t) = \mathrm{e}^{2(t-t_0)}\left(a^2(t_0) + 2\int_{t_0}^{t}\mathrm{d}s\,\mathrm{e}^{2(t_0-s)}\frac{\boldsymbol{a}\cdot\boldsymbol{x}}{r^3}\right). \tag{11}$$

Since $a^2 \geqslant 0$, the above equation implies

$$a^2(t_0) \geqslant -2 \int_{t_0}^{t} \mathrm{d}s\, \mathrm{e}^{2(t_0-s)} \frac{\boldsymbol{a}\cdot\boldsymbol{x}}{r^3}. \tag{12}$$

Definition 1. *A solution $\boldsymbol{x} : [\bar{t}, +\infty) \to \mathbb{R}^3$ of equation (3) is said to be 'runaway' if $r \to \infty$ and the acceleration a diverges exponentially for $t \to +\infty$.*

Proposition 1. *Let $\boldsymbol{x} : [\bar{t}, +\infty) \to \mathbb{R}^3$ be a solution of equation (3). If there is a time t_0 for which $\dot{r}(t_0) \geqslant 0$ and $\boldsymbol{v}(t_0)\cdot\boldsymbol{a}(t_0) > 0$, then the solution is runaway.*

Proof. Let us introduce the unit vector $\boldsymbol{n} = \boldsymbol{v}(t_0)/v(t_0)$. Taking the scalar product with $\boldsymbol{n}$ of both members of equation (5) we get

$$\boldsymbol{a}(t)\cdot\boldsymbol{n} = \mathrm{e}^{t-t_0}\boldsymbol{a}(t_0)\cdot\boldsymbol{n} + \int_{t_0}^{t} \mathrm{e}^{t-s}\frac{\boldsymbol{x}(s)\cdot\boldsymbol{n}}{r^3(s)}\,\mathrm{d}s. \tag{13}$$

We have by hypothesis $\boldsymbol{a}(t_0)\cdot\boldsymbol{n} > 0$ and $\boldsymbol{x}(t_0)\cdot\boldsymbol{n} = \dot{r}(t_0)r(t_0)/v(t_0) \geqslant 0$. One can see from equation (13) that if the integrand on the rhs is $\geqslant 0$ for all $s \geqslant t_0$, then $\boldsymbol{a}(t)\cdot\boldsymbol{n}$, and therefore also $\boldsymbol{x}(t)\cdot\boldsymbol{n}$, diverge exponentially for $t \to +\infty$. This in turn implies that both $a(t)$ and $r(t)$ are also exponentially divergent. Therefore, the solution is certainly runaway, unless there exists some $t_1 > t_0$ for which $\boldsymbol{x}(t_1)\cdot\boldsymbol{n} < 0$. We are now going to show that this cannot occur. In fact, since $\boldsymbol{v}(t_0)\cdot\boldsymbol{n} = v(t_0) > 0$, there should be in that case a t_2, with $t_0 < t_2 < t_1$, such that $\boldsymbol{v}(t_2)\cdot\boldsymbol{n} = 0$ and $\boldsymbol{v}(t)\cdot\boldsymbol{n} > 0$ for every $t \in [t_0, t_2)$. This implies $\boldsymbol{x}(t)\cdot\boldsymbol{n} \geqslant 0$ and therefore, using again (13), $\boldsymbol{a}(t)\cdot\boldsymbol{n} > 0$ for every $t \in [t_0, t_2]$. From this, however, it follows:

$$\boldsymbol{v}(t_2)\cdot\boldsymbol{n} = \boldsymbol{v}(t_0)\cdot\boldsymbol{n} + \int_{t_0}^{t_2} \boldsymbol{a}(t)\cdot\boldsymbol{n}\,\mathrm{d}t > 0$$

in contradiction with $\boldsymbol{v}(t_2)\cdot\boldsymbol{n} = 0$. □

We are now ready to prove our main theorem.

Theorem 1. *If $\boldsymbol{x} : [\bar{t}, +\infty) \to \mathbb{R}^3$ is a solution of equation (3), then $\lim_{t\to+\infty} r(t) = +\infty$. Moreover, if the solution is nonrunaway, then $\int_{\bar{t}}^{+\infty} a^2\,\mathrm{d}t < +\infty$ and $\lim_{t\to+\infty} \boldsymbol{a}(t) = 0$.*

Proof. We shall distinguish between two possibilities.

Case 1. $\int_{\bar{t}}^{+\infty} a^2\,\mathrm{d}t < +\infty$, so that $\lim_{t\to+\infty} E(t) = \bar{E} > -\infty$. We shall see that in this case the particle is scattered away with vanishing asymptotic acceleration.

Let us first make the hypothesis that the trajectory is bounded, i.e. there exists M such that $r(t) < M$ for every t. According to lemma 2 , we have then $\lim_{t\to+\infty} \boldsymbol{v} = 0$. Furthermore we have $\limsup_{t\to\infty} r > 0$. In fact, if $\lim_{t\to\infty} r = 0$, from (6) it would follow that $\lim_{t\to\infty} \boldsymbol{a}\cdot\boldsymbol{v} = -\infty$, and therefore $\lim_{t\to\infty} a = \infty$, in contradiction with $\int_{\bar{t}}^{+\infty} a^2\,\mathrm{d}t < +\infty$. Let us now fix $\bar{r}$ such that $0 < \bar{r} < \limsup_{t\to\infty} r$. For any arbitrarily large T there exists t_0 such that $r(t_0) > \bar{r}$ and $\sup_{t\geqslant t_0} v < \bar{r}/2T$. Introducing the unit vector $\boldsymbol{n} = \boldsymbol{x}(t_0)/r(t_0)$, we then have for $t_0 \leqslant t \leqslant t_0 + T$

$$\bar{r} - \boldsymbol{n}\cdot\boldsymbol{x}(t) < \boldsymbol{n}\cdot[\boldsymbol{x}(t_0) - \boldsymbol{x}(t)] \leqslant |\boldsymbol{x}(t_0) - \boldsymbol{x}(t)| \leqslant \int_{t_0}^{t} v(s)\,\mathrm{d}s < \frac{\bar{r}}{2}$$

whence $\boldsymbol{n}\cdot\boldsymbol{x}(t) > \bar{r}/2$. From (3) it then follows

$$\boldsymbol{n}\cdot[\dot{\boldsymbol{a}}(t) - \boldsymbol{a}(t)] = \frac{\boldsymbol{n}\cdot\boldsymbol{x}(t)}{r^3(t)} > \frac{\bar{r}}{2M^3}.$$

We can now apply lemma 3 to the function $f(t) \equiv \boldsymbol{n} \cdot \boldsymbol{a}(t)$, obtaining

$$\int_{t_0}^{t_0+T} a^2\,\mathrm{d}t \geqslant \int_{t_0}^{t_0+T} f^2\,\mathrm{d}t \geqslant \left(\frac{\bar{r}}{2M^3}\right)^2 (T-2).$$

We have thus arrived at a contradiction, since owing to the arbitrariness of T the above inequality would imply $\int_{\bar{t}}^{+\infty} a^2\,\mathrm{d}t = +\infty$. We can so deduce that the trajectory is unbounded, i.e. $\limsup_{t\to+\infty} r = +\infty$.

The thesis of the theorem is not yet proved, since in principle it is still possible that $\liminf_{t\to+\infty} r < +\infty$. We shall show, however, with a suitable modification of the above argument, that this assumption also leads to a contradiction. Let us take $L > \liminf_{t\to+\infty} r$. Then for any $\varepsilon > 0$ there is t_0 such that $E(t_0) - \bar{E} = \int_{t_0}^{+\infty} a^2\,\mathrm{d}t < \varepsilon$ and $r(t_0) < L$. Since the trajectory has been proved to be unbounded, by lemma 1 there must also be $t_1 > t_0$ such that $r(t_1) = L, \dot{r}(t_1) \geqslant 0$. We then have $\boldsymbol{v}(t_1) \cdot \boldsymbol{a}(t_1) \leqslant 0$, since otherwise according to proposition 1 the trajectory would be runaway. It follows from (6) that

$$\frac{1}{2}v^2(t_1) \leqslant \frac{1}{r(t_1)} + \bar{E} + \int_{t_1}^{+\infty} a^2\,\mathrm{d}t < \frac{1}{L} + \bar{E} + \varepsilon. \tag{14}$$

Since ε and $1/L$ can be taken arbitrarily small, the above equation implies in particular that $\bar{E} \geqslant 0$. Let us now fix $T > 0$. For every $t \in [t_1, t_1 + T]$ we have

$$|\boldsymbol{v}(t) - \boldsymbol{v}(t_1)| \leqslant \int_{t_1}^{t} a\,\mathrm{d}t' \leqslant \left(\int_{t_1}^{t} a^2\,\mathrm{d}t'\right)^{1/2} \left(\int_{t_1}^{t} \mathrm{d}t'\right)^{1/2} < \sqrt{\varepsilon T}.$$

Hence, introducing the vector

$$\boldsymbol{y}(t) = \boldsymbol{x}(t) - \boldsymbol{x}(t_1) - (t - t_1)\,\boldsymbol{v}(t_1) = \int_{t_1}^{t} [\boldsymbol{v}(s) - \boldsymbol{v}(t_1)]\,\mathrm{d}s$$

we have

$$|\boldsymbol{y}(t)| \leqslant \int_{t_1}^{t} |\boldsymbol{v}(s) - \boldsymbol{v}(t_1)|\mathrm{d}s < \varepsilon^{1/2}T^{3/2}. \tag{15}$$

Using the triangular inequality we derive from (14) and (15)

$$r(t) = |\boldsymbol{x}(t_1) + (t - t_1)\boldsymbol{v}(t_1) + \boldsymbol{y}(t)| < L + T\sqrt{2(1/L + \bar{E} + \varepsilon)} + \varepsilon^{1/2}T^{3/2} \equiv \bar{r}(\varepsilon).$$

Furthermore, introducing the unit vector $\boldsymbol{n} = \boldsymbol{x}(t_1)/L$, we have $\boldsymbol{v}(t_1) \cdot \boldsymbol{n} = \dot{r}(t_1) \geqslant 0$ and therefore, using again (15),

$$\boldsymbol{x}(t) \cdot \boldsymbol{n} = [\boldsymbol{x}(t_1) + (t - t_1)\boldsymbol{v}(t_1) + \boldsymbol{y}(t)] \cdot \boldsymbol{n} > L - \varepsilon^{1/2}T^{3/2}.$$

We have then for $t \in [t_1, t_1 + T]$

$$\boldsymbol{n} \cdot [\dot{\boldsymbol{a}}(t) - \boldsymbol{a}(t)] = \frac{\boldsymbol{n} \cdot \boldsymbol{x}(t)}{r^3(t)} > \frac{L - \varepsilon^{1/2}T^{3/2}}{[\bar{r}(\varepsilon)]^3} \equiv B(\varepsilon).$$

Since L and T can be fixed independently of ε, and

$$\lim_{\varepsilon\to 0} B(\varepsilon) = \frac{L}{[L + T\sqrt{2(1/L + \bar{E})}]^3} \equiv \bar{B} > 0$$

for sufficiently small ε we have $B(\varepsilon) > \bar{B}/2 > 0$. Therefore, applying lemma 3 to the function $f(t) \equiv \boldsymbol{n} \cdot \boldsymbol{a}(t)$, we get

$$\int_{t_1}^{t_1+T} a^2\,\mathrm{d}t \geqslant \int_{t_1}^{t_1+T} f^2\,\mathrm{d}t \geqslant \frac{\bar{B}^2}{4}[T - 2\tanh(T/2)]$$

which for ε smaller than the rhs is in contradiction with $\int_{t_0}^{+\infty} a^2\,\mathrm{d}t < \varepsilon$. We can thus conclude that $\liminf_{t\to+\infty} r = \lim_{t\to+\infty} r = +\infty$.

From this result it follows in particular that the integral on the rhs of (5) has a finite limit for $t \to \infty$. Since in the case considered a cannot diverge exponentially, equation (5) implies

$$\boldsymbol{a}(t_0) = -\int_{t_0}^{+\infty} \mathrm{e}^{t_0-s}\,\frac{\boldsymbol{x}(s)}{r^3(s)}\,\mathrm{d}s$$

for all t_0, and so $\lim_{t\to+\infty} \boldsymbol{a}(t) = 0$.

Case 2. $\int_{\bar{t}}^{+\infty} a^2\,\mathrm{d}s = +\infty$, so that $\lim_{t\to+\infty} E(t) = -\infty$. We shall prove that this situation always corresponds to a runaway solution.

Let us first suppose that a solution is unbounded. Then, having taken t_0 such that $E(t_0) < 0$, by lemma 1 there must be $t > t_0$ such that $r(t) > -1/E(t_0)$ and $\dot{r}(t) \geqslant 0$. But then, using (6) and the fact that E is a nonincreasing function, one obtains

$$\boldsymbol{v}(t)\cdot\boldsymbol{a}(t) \geqslant \frac{v^2(t)}{2} - \frac{1}{r(t)} - E(t_0) > 0.$$

Therefore, the solution turns out to be runaway upon application of proposition 1.

We further observe that, according to (10), if $F(t_0) > 0$ for some t_0, then $\boldsymbol{a}\cdot\boldsymbol{x}$ diverges exponentially to $+\infty$ for $t \to +\infty$, and the identity $\ddot{r}r = \boldsymbol{a}\cdot\boldsymbol{x} + v^2 - \dot{r}^2 \geqslant \boldsymbol{a}\cdot\boldsymbol{x}$ implies that the same is true for the function $\ddot{r}r$. One has then $\dot{r}r = \int^t (\ddot{r}r + \dot{r}^2)\,\mathrm{d}t' \to +\infty$ for $t \to +\infty$, and so $r^2 = 2\int^t \dot{r}r\,\mathrm{d}t' \to +\infty$. This implies that the solution is unbounded and so, as we have just seen, runaway.

On the grounds of these results, from now on we shall only consider (hypothetical) bounded solutions for which $F(t) \leqslant 0$ for every t. We then obtain from (12) for $t' > t$

$$a^2(t) \geqslant 2\int_t^{t'} \mathrm{d}s\,\mathrm{e}^{2(t-s)}\,\frac{W(s)}{r^3(s)} \tag{16}$$

where $W \equiv -E$. Introducing also the nonincreasing function $M(t) = \sup_{s\geqslant t} r(s)$, we obtain from (16) taking the limit for $t' \to +\infty$

$$\frac{\mathrm{d}W(t)}{\mathrm{d}t} = a^2 > \frac{2W(t)}{M^3(t)}\int_t^{+\infty} \mathrm{d}s\,\mathrm{e}^{2(t-s)} = \frac{W(t)}{M^3(t)}. \tag{17}$$

For $W(t) > 0$ and $s > t$ we then have

$$W(s) > W(t)\exp\left[\int_t^s \frac{\mathrm{d}u}{M^3(u)}\right] \geqslant W(t)\exp\frac{s-t}{M^3(t)} \tag{18}$$

which when substituted into (16) provides

$$a^2(t) \geqslant \frac{2W(t)}{M^3(t)}\int_t^{+\infty} \mathrm{d}s\,\exp[(t-s)(2-M^{-3}(t))]. \tag{19}$$

From equations (18) and (17) one can see that both W and a^2 diverge exponentially for $t \to +\infty$. Furthermore, since inequality (19) implies that the integral at the rhs be convergent, we have $2 - M^{-3}(t) > 0$, and so $\lim_{t\to+\infty} M(t) = \limsup_{t\to+\infty} r(t) \geqslant 1/2^{1/3}$. Now take any t large enough so that, say, $E(t) < -3$. If $r(t) < 1/2^{1/3}$, then by lemma 1 there must be $t' > t$ such that $r(t') > 1/3^{1/3}$ and $\dot{r}(t') \geqslant 0$. Therefore

$$\boldsymbol{v}(t')\cdot\boldsymbol{a}(t') > \frac{v^2(t')}{2} - \frac{1}{r(t')} - E(t) > \frac{v^2(t')}{2} - 3^{1/3} + 3 > 0$$

and again one can make use of proposition 1 to deduce that the solution is runaway. It follows that, in order for a solution to be nonrunaway, there must exist t_0 such that $r(t) \geqslant 1/2^{1/3}$ for every $t \geqslant t_0$.

Let us then suppose that such a t_0 exists. Since we have proved that $\lim_{t\to+\infty} a = +\infty$, there must be $t_1 > t_0$ such that $a(t_1) > 1 + 2^{2/3}$. Introducing the unit vector $\boldsymbol{n} = \boldsymbol{a}(t_1)/a(t_1)$, for $t \geqslant t_0$ we have $|\boldsymbol{n}\cdot\boldsymbol{x}(t)|/r^3(t) \leqslant 1/r^2(t) \leqslant 2^{2/3}$. Therefore for $t \geqslant t_1$, using (5) we get

$$\boldsymbol{a}(t)\cdot\boldsymbol{n} \geqslant \mathrm{e}^{t-t_1}\boldsymbol{a}(t_1)\cdot\boldsymbol{n} - 2^{2/3}(\mathrm{e}^{t-t_1} - 1) > \mathrm{e}^{t-t_1} + 2^{2/3}$$

so that $\lim_{t\to+\infty} \boldsymbol{n}\cdot\boldsymbol{x}(t) = +\infty$ and the solution is runaway also in this case. This completes the proof of the theorem. □

3. Comments

Since runaway solutions are considered to be unphysical, from the above theorem it follows that all acceptable solutions of equation (3) are of scattering type. It is not known in general whether, given a pair of initial values of position and velocity, there exist some initial values of the acceleration for which the corresponding solution is nonrunaway. It is, however, possible to establish a simple condition that a physically acceptable set of initial data is bound to satisfy. In fact for a nonrunaway solution, since we have proved that $\lim_{t\to+\infty} \boldsymbol{a}(t) = 0$, recalling (6) we have

$$\bar{E} = \lim_{t\to+\infty} E(t) = \lim_{t\to+\infty} \boldsymbol{v}\cdot(\boldsymbol{v}/2 - \boldsymbol{a}) = \lim_{t\to+\infty} v^2/2 \geqslant 0.$$

But since E is a nonincreasing function of time, this necessarily implies that at all times one has $E \geqslant 0$. One can then immediately observe that in general, for a given r, the acceleration $\boldsymbol{a}$ has to diverge for vanishing $\boldsymbol{v}$. In particular, no nonrunaway solutions can exist for which at a given time one has $\boldsymbol{v} = 0$, independently of the distance from the centre of force. Note, however, that the usual sum of kinetic and potential energy $E_c = v^2/2 - 1/r$ might temporarily assume negative values, provided the acceleration simultaneously acquires a sufficiently large component antiparallel to the velocity. Of course, this can be possible because E_c is not conserved for a particle obeying equation (3). Other interesting phenomena related to the same fact have already been revealed in the numerical study of the one-dimensional Lorentz–Dirac equation in the presence of a potential barrier. They include the classical analogues of well-known quantum phenomena such as weak reflection and tunnel effect [7, 8].

Let us now consider for instance an initial condition which, in the absence of radiation reaction, would correspond to a circular orbit, so that $v^2 = 1/r$. If we denote with a_T the tangential component of the acceleration (with verse opposite to the velocity), and with $a_C = 1/r^2$ the centripetal acceleration that would correspond to the Coulomb force alone, it is easy to see that the condition $E \geqslant 0$ implies $a_T/a_C \geqslant 1/2v^3$, which in the customary units used in equation (2) becomes $a_T/a_C \geqslant 3Z/4v^3$. One can see therefore that, at least in the nonrelativistic case $v \ll 1$, the tangential acceleration due to radiation reaction has to be largely predominant in order that nonrunaway solutions may possibly exist.

It would be, of course, extremely interesting to extend the conclusions of theorem 1 to the relativistic Lorentz–Dirac equation and to the case of two (or more) massive interacting particles, instead of a fixed centre of force. It has in fact been proved that in such a case, when the independent degrees of freedom of the electromagnetic field are duly taken into account, the dynamics of particles and field can be treated as an infinite-dimensional conservative Hamiltonian system [9]. If results similar to theorem 1 were found to hold, since scattering solutions clearly have positive total energy (when in the definition of energy one subtracts the rest-mass of the particles involved), this would seem to indicate that there do not exist negative energy states in the physical (i.e. nonrunaway) phase-space of the associated Hamiltonian.

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